The Unmasking of Thermal Goldstone Bosons

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Abstract

The problem of extracting the modes of Goldstone bosons from a thermal background is reconsidered in the framework of relativistic quantum field theory. It is shown that in the case of spontaneous breakdown of an internal bosonic symmetry a recently established decomposition of thermal correlation functions contains certain specific contributions which can be attributed to a particle of zero mass.

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The relation between the spontaneous breakdown of symmetries and the structure of the energy–momentum spectrum is known to be more subtle for thermal states than in the case of the vacuum. According to the thermal version of the Goldstone theorem, there appears a discrete zero energy mode in current–field correlation functions whenever the symmetry corresponding to the current is spontaneously broken, see for example [1]. But, in contrast to the vacuum case, there need not hold a sharp energy–momentum dispersion law which would allow one to identify clearly the Goldstone particles in the usual way. As a matter of fact, the existence of such a law would imply that there is no interaction [2].

The fact that there appear in general only low–energy artifacts of Goldstone particles in thermal correlation functions can easily be understood in heuristic physical terms: the particles collide with large probability with other constituents of the state and thereby change their energy and momentum. Hence discrete (δ–function) contributions in the correlation functions corresponding to events where a particle excitation remains unaffected by the thermal background are dynamically suppressed. It is only at zero energy where such contributions can survive since soft massless particles do not participate in any reactions according to well–known low energy theorems (cf. [3] and references quoted therein).

There appears yet another complication in the case of spontaneously broken geometric symmetries, such as Lorentz transformations, and supersymmetries. As was first pointed out in [4], the discrete zero energy modes are in general not

affiliated with a particle in these cases but are due to particle–hole pairs. In view of these facts the foundations for a particle interpretation of the Goldstone modes in thermal quantum field theory remained unsettled to date.

It is the aim of the present article to show that in the case of spontaneously broken internal symmetries one can identify the Goldstone modes in thermal correlation functions by a novel method which reveals their particle nature in a clearcut manner. Our approach is based on a general resolution of thermal correlation functions which seems to be better adapted to the dissipative effects of a thermal background than mere Fourier analysis. This resolution has recently been established in the framework of relativistic quantum field theory for thermal correlation (two–point) functions of arbitrary pairs of local field operators [5].

Within the present context we are particularly interested in correlation functions involving the zero component of a conserved current j_μ and a field ϕ,

$$\langle\, j_0(x)\phi(y)\,\rangle_\beta = \lim \frac{1}{Z} \mathrm{Tr}\, e^{-\beta H}\, j_0(x)\phi(y). \tag{1}$$

Here $\beta > 0$ is the inverse temperature and the right hand side of relation (1) is a reminder of the fact that the correlation functions are obtained by performing the Gibbs construction in a finite volume and proceeding to the thermodynamic limit. Assuming that the state $\langle\,\cdot\,\rangle_\beta$ is invariant under spacetime translations and that $j_\mu(x)$ and $\phi(y)$ commute at spacelike distances, $(x-y)^2 < 0$, we can apply the results in [5], giving the following resolution of the correlation functions (where we have put $y = 0$ in order to simplify the notations),

$$\langle\, j_0(x)\phi(0)\,\rangle_\beta = \int_0^\infty dm\, (D_\beta^{(+)}(\boldsymbol{x}, m)\, \partial_0 W_\beta^{(0)}(x, m) + D_\beta^{(-)}(\boldsymbol{x}, m) W_\beta^{(0)}(x, m)). \tag{2}$$

Here $(x_0, \boldsymbol{x})$ denote the privileged time and space coordinates of x in the Lorentz system fixed by the thermal state, ∂_0 is the time derivative, $D_\beta^{(\pm)}(\boldsymbol{x}, m)$ are tempered distributions (depending on the underlying theory) and

$$W_\beta^{(0)}(x, m) = (2\pi)^{-3} \int d^4p\, \varepsilon(p_0)\delta(p^2 - m^2)(1 - e^{-\beta p_0})^{-1} e^{-ipx} \tag{3}$$

is the two–point function of a free scalar field of mass m in a thermal equilibrium state at inverse temperature β. The corresponding resolution of the commutator function is given by

$$\langle\, [j_0(x), \phi(0)]\,\rangle_\beta = \int_0^\infty dm\, (D_\beta^{(+)}(\boldsymbol{x}, m)\, \partial_0 \Delta(x, m) + D_\beta^{(-)}(\boldsymbol{x}, m)\Delta(x, m)), \tag{4}$$

where $\Delta(x, m)$ denotes the Pauli–Jordan distribution

$$\Delta(x, m) = (2\pi)^{-3} \int d^4p\, \varepsilon(p_0)\delta(p^2 - m^2) e^{-ipx}. \tag{5}$$

Similar resolutions hold for the time–ordered, advanced and retarded functions.

As proved in [5], relations (2) and (4) are consequences of the basic principle of relativistic causality and provide a natural generalization of the Källén–Lehmann

representation in the vacuum theory to the case of thermal equilibrium states. This relationship becomes even more obvious if one also takes into account the analyticity properties of thermal correlation functions in the complex spacetime variables which follow from the relativistic form of the energy–momentum spectrum in the vacuum sector (relativistic KMS–condition [6]). We will make use of this additional structure in the subsequent discussion of our results.

The representations (2) and (4) show that, in complete analogy to the vacuum case, the relativistic thermal correlation functions can be resolved into a superposition of free field thermal correlation functions of mass $m \geq 0$. But whereas the coefficients $D_\beta^{(\pm)}(\boldsymbol{x}, m)$ are constant in $\boldsymbol{x}$ in the vacuum theory they exhibit in the case of thermal states in general a non–trivial $\boldsymbol{x}$–dependence, describing the dissipative effects of the thermal background on excitations induced by the fields. As these effects give rise to the damping of amplitudes, the coefficients $D_\beta^{(\pm)}(\boldsymbol{x}, m)$ were called *damping factors* in [7]. Note that the multiplicative action of the damping factors on the free field correlation functions amounts to a convolution in momentum space which smoothes out in general the discrete mass–shell contributions of stable particles, in agreement with the general statements proved in [2]. However, as was shown in [7], such particles may still be identified by discrete mass–contributions appearing in the resolutions (2) and (4) in configuration space. It is this observation that we intend to use for the identification of Goldstonean massless particles associated with spontaneous symmetry breaking in thermal equilibrium states.

Before we can apply our representations to the problem of spontaneous symmetry breaking, we have to comment on the precise mathematical meaning of relations (2) and (4), which may not be completely obvious in view of the singular nature of the quantities involved. The left hand side of relation (4), say, is defined in the sense of tempered distributions, i.e., becomes meaningful if integrated with a test function. It suffices for our purposes to consider test functions of the form $f(x_0)g(\boldsymbol{x})$, where f, g have compact support. For the specification of the right hand side of relation (4) we make use of the fact that

$$h^{(\pm)}(\boldsymbol{x}, m) \doteq \int dx_0\, f(x_0)\, \partial^{(\pm)}\Delta(x, m), \tag{6}$$

(where $\partial^{(+)} \doteq \partial_0, \partial^{(-)} \doteq 1$) are test functions in the variables $\boldsymbol{x}, m$ for any choice of the test function f. The rigorous version of relation (4) can then be presented in the form

$$\begin{aligned} &\int d^4x\, f(x_0)g(\boldsymbol{x})\,\langle\, [j_0(x), \phi(0)]\,\rangle_\beta = \qquad\qquad (7) \\ &\quad = \int d^3\boldsymbol{x} \int dm\, (D_\beta^{(+)}(\boldsymbol{x}, m)g(\boldsymbol{x})h^{(+)}(\boldsymbol{x}, m) + D_\beta^{(-)}(\boldsymbol{x}, m)g(\boldsymbol{x})h^{(-)}(\boldsymbol{x}, m)), \end{aligned}$$

where the right hand side of (7) is meaningful since $D_\beta^{(\pm)}(\boldsymbol{x}, m)$ are tempered distributions with support in $\mathbf{R}^3 \times \mathbf{R}_+$. In a similar fashion one can give a precise meaning to relation (2).

Let us now turn to the discussion of the symmetry transformation induced by the current j_μ. The charge operator corresponding to this current (if it exists) is

defined as a suitable limit of the operators

$$Q_R \doteq \int d^4x \, f(x_0) g(\boldsymbol{x}/R) \, j_0(x) \tag{8}$$

for $R \to \infty$, where g is any test function which is equal to 1 in the unit ball about the origin of configuration space and f is usually normalized according to $\int dx_0 \, f(x_0) = 1$. (In order to display the role of f in the subsequent argument we do not impose here the latter condition, however.) The operators Q_R are thus the appropriately regularized charge operators for finite spatial volume.

It follows from the spacelike commutativity of $j_\mu(x), \phi(y)$ and current conservation that for sufficiently large R (depending on the choice of the function f) there holds

$$\langle \, [Q_R, \phi(0)] \, \rangle_\beta = q \int dx_0 \, f(x_0), \tag{9}$$

where q is some constant which does not depend on the choice of f, g within the above limitations. This result can be established by the same arguments as in the vacuum theory [8], so we do not need to reproduce it here. The symmetry corresponding to the current j_μ is said to be spontaneously broken in the state $\langle \, \cdot \, \rangle_\beta$, if $q \neq 0$ for some field ϕ. Then the limit of the operators Q_R does not define the generator of a unitary group which induces the symmetry transformation and leaves the state $\langle \, \cdot \, \rangle_\beta$ invariant.

As is well known, Eqs. (8) and (9) imply that the Fourier transform of the commutator function $\langle \, [j_0(x), \phi(0)] \, \rangle_\beta$ is equal to $(q/2\pi) \, \delta(p_0)$ for spatial momentum $\boldsymbol{p} = 0$. But, in contrast to the vacuum case, the occurrence of a δ–singularity on the full light cone $p^2 = 0$ is not implied by this fact. So there is *a priori* no basis for the conventional momentum-space interpretation of these zero–energy modes as relativistic particles. As already mentioned, there are examples just outside the present framework (based on currents $j_\mu(x)$ which depend explicitly on the spacetime coordinates x) where these modes cannot be associated with a particle [4]. However, for the local covariant currents considered here, we are able to show that if relation (9) holds for some $q \neq 0$, there appear (in a generic way) certain specific discrete contributions in the damping factor $D_\beta^{(+)}(\boldsymbol{x}, m)$ in our representation (2) which can be attributed to a Goldstone boson. These particles can thus be "unmasked" in configuration space.

Turning to the proof of this statement we first note that for odd test functions f the right hand side of relation (9) vanishes. So we may restrict attention to even f and proceed, by combining relations (7) to (9), to

$$\int d^3\boldsymbol{x} \int dm \, D_\beta^{(+)}(\boldsymbol{x}, m) \, g(\boldsymbol{x}/R) h^{(+)}(\boldsymbol{x}, m) = q \int dx_0 \, f(x_0) \tag{10}$$

which holds for sufficiently large R. Since $g(0) = 1$, the sequence of test functions $g(\boldsymbol{x}/R) h^{(+)}(\boldsymbol{x}, m)$ converges for $R \to \infty$ to $h^{(+)}(\boldsymbol{x}, m)$ (in the appropriate topology), hence equation (10) becomes in this limit

$$\int d^3\boldsymbol{x} \int dm \, D_\beta^{(+)}(\boldsymbol{x}, m) h^{(+)}(\boldsymbol{x}, m) = q \int dx_0 \, f(x_0). \tag{11}$$

The constraints imposed on $D_\beta^{(+)}(\boldsymbol{x},m)$ by the latter equation are more transparent in momentum space. There one gets, bearing in mind the definition (6) of $h^{(+)}(\boldsymbol{x},m)$,

$$\int d^3\boldsymbol{p} \int dm\, \widetilde{D}_\beta^{(+)}(\boldsymbol{p},m)\widetilde{f}(\sqrt{\boldsymbol{p}^2+m^2}) = iq\,(2\pi)^{3/2}\widetilde{f}(0), \tag{12}$$

where $\widetilde{f}(p_0)$ denotes the Fourier transform of $f(x_0)$ and $\widetilde{D}_\beta^{(+)}(\boldsymbol{p},m)$ the (partial) Fourier transform of $D_\beta^{(+)}(\boldsymbol{x},m)$ with respect to the spatial variables $\boldsymbol{x}$.

Relation (12) holds for any choice of the (even) test function f. The corresponding functions $\widetilde{f}(\sqrt{\boldsymbol{p}^2+m^2})$ on the space $\mathbf{R}^4$ spanned by $\boldsymbol{p}, m$ exhaust the set of all spherically symmetric test functions on this space. Hence we infer from (12) that in the (unique) decomposition of the distribution $\widetilde{D}_\beta^{(+)}(\boldsymbol{p},m)$ into its spherically symmetric part in $(\boldsymbol{p},m)$–space and the remainder, the symmetric part consists of a multiple of the δ–function,

$$\widetilde{D}_\beta^{(+)}(\boldsymbol{p},m) = iq\,(2\pi)^{3/2}\delta(\boldsymbol{p})\delta(m) + \widetilde{R}_\beta^{(+)}(\boldsymbol{p},m), \tag{13}$$

and going back to configuration space we arrive at

$$D_\beta^{(+)}(\boldsymbol{x},m) = iq\,\delta(m) + R_\beta^{(+)}(\boldsymbol{x},m). \tag{14}$$

This relation provides some evidence to the effect that in $D_\beta^{(+)}(\boldsymbol{x},m)$ there appears generically a discrete zero mass contribution if the symmetry is spontaneously broken. Yet this idea requires some further analysis since the remainder in (14) could contain additional singular terms which completely screen the effects of the delta function; in other words, relation (14) does not establish a decomposition of the damping factor into contributions of different degree of singularity. In order to clarify this point we consider in the following the physically interesting case where $dm\, D_\beta^{(+)}(\boldsymbol{x},m)$ is a (complex) measure in m which is regular in $\boldsymbol{x}$. The latter property follows from the relativistic KMS–condition [5] and the former one may be expected to hold quite generally, similarly to the case of the "weight functions" in the Källén–Lehmann representation of vacuum correlation functions, cf. [7]. We recall that according to standard theorems on the decompositon of measures the statement that a measure has a discrete contribution has a clearcut mathematical meaning. Moreover, discrete contributions are the most prominent singularities which can occur.

In order to establish rigorously from equation (12) that such a discrete zero mass contribution is present in $D_\beta^{(+)}(\boldsymbol{x},m)$ in generic cases, let us make the additional physically motivated assumption that the damping factor decreases for increasing $|\boldsymbol{x}|$. More precisely, presenting $D_\beta^{(+)}(\boldsymbol{x},m)$ as the Radon–Nikodym derivative of a piecewise continuous, bounded function,

$$D_\beta^{(+)}(\boldsymbol{x},m) = \partial_m C_\beta^{(+)}(\boldsymbol{x},m), \tag{15}$$

let us assume that $|C_\beta^{(+)}(\boldsymbol{x},m)|$ is monotonically decreasing in $|\boldsymbol{x}|$ for fixed $\boldsymbol{x}/|\boldsymbol{x}|$ and m. Such a behavior of the damping factors may be expected in the presence of dissipative effects of a spatially homogeneous thermal background [7].

By writing equation (12) with the choice of Gaussian functions $\widetilde{f}(p_0) = e^{-\lambda^2 p_0^2/2}$, one obtains after a straightforward computation

$$
\begin{aligned}
iq\,(2\pi)^{3/2} &= \lim_{\lambda\to\infty} \int d^3\boldsymbol{p} \int dm\, \widetilde{D}_\beta^{(+)}(\boldsymbol{p}, m)\, e^{-\lambda^2(\boldsymbol{p}^2+m^2)/2} \\
&= \lim_{\lambda\to\infty} \int d^3\boldsymbol{x} \int dm\, m\, C_\beta^{(+)}(\lambda\boldsymbol{x}, \lambda^{-1}m)\, e^{-(\boldsymbol{x}^2+m^2)/2}.
\end{aligned}
\tag{16}
$$

It therefore follows (in view of the dominated convergence theorem, which governs the interchange of limits and integrations) that the function $C_\beta^{(+)}(\boldsymbol{x}, m)$ cannot exhibit a "trivial scaling behavior" of the form $\lim_{\lambda\to\infty} C_\beta^{(+)}(\lambda\boldsymbol{x}, \lambda^{-1}m) = 0$ for all $\boldsymbol{x}$ and fixed $m > 0$ if $q \neq 0$. On the other hand, the monotonicity properties of $C_\beta^{(+)}(\boldsymbol{x}, m)$ imply that for $\lambda \geq 1$ there holds $|C_\beta^{(+)}(\boldsymbol{x}, \lambda^{-1}m)| \geq |C_\beta^{(+)}(\lambda\boldsymbol{x}, \lambda^{-1}m)|$. Proceeding to the limit $\lambda \to \infty$ we therefore conclude that

$$
\lim_{m\downarrow 0} C_\beta^{(+)}(\boldsymbol{x}, m) \not\equiv 0. \tag{17}
$$

Since $C_\beta^{(+)}(\boldsymbol{x}, m)$ vanishes for negative m this shows that the latter function is discontinuous in m at $m = 0$ and consequently (in view of (15)) $D_\beta^{(+)}(\boldsymbol{x}, m)$ has a discrete zero mass contribution of the form $C_\beta(\boldsymbol{x})\,\delta(m)$, where $C_\beta(\boldsymbol{x})$ denotes the limit in (17). (It can be seen that the same conclusion also holds under considerably weaker assumptions.)

There is another interesting consequence of relation (16) with regard to the spatial behavior of the damping factors. Because of the dissipative effects already mentioned one expects that the function $C_\beta^{(+)}(\boldsymbol{x}, m)$ tends to 0 for large $|\boldsymbol{x}|$ for almost all m. We now claim that this decay has to be slow in the presence of spontaneous symmetry breaking. For if there holds for large $|\boldsymbol{x}|$ and small m

$$
|m^\delta\, C_\beta^{(+)}(\boldsymbol{x}, m)| \leq const\, |\boldsymbol{x}|^{-\varepsilon} \tag{18}
$$

for some $\varepsilon > \delta \geq 0$, it follows from (16) that $q = 0$. This result is in accord with the physical picture that sufficiently strong dissipative effects destroy the long range order in thermal states and thereby lead to a restoration of symmetries.

Note however that the bound in (18) does not exclude the presence of a discontinuity of $C_\beta^{(+)}(\boldsymbol{x}, m)$ at $m = 0$. This shows that, in contrast to the vacuum case, there may appear discrete massless contributions in the damping factors of current–field correlation functions without spontaneous breakdown of symmetries and this occurs as an effect of the strength of the damping. This phenomenon is consistent with the following scenario which is of interest from a physical viewpoint: if some spontaneously broken symmetry, which is accompanied by a Goldstone particle, is restored at high temperatures, this particle need not cease to exist. From a more technical viewpoint, based on the general form (15) of our damping factor $D_\beta^{(+)}(\boldsymbol{x}, m)$, what matters for the breakdown of symmetries is the scaling behavior of $C_\beta^{(+)}(\boldsymbol{x}, m)$ whose non–vanishing character can be considered as a criterion for symmetry breaking.

To summarize, we have seen in the preceding analysis that in the case of spontaneously broken internal symmetries there appear generically in the representation (2) of current–field correlation functions damping factors $D_\beta^{(+)}(\boldsymbol{x}, m)$ which

(a) contain a discrete (in the sense of measures) zero–mass contribution and

(b) are slowly decreasing in $|\boldsymbol{x}|$ for small values of m.

So these damping factors coincide locally with the Källén–Lehmann weights appearing in the case of spontaneous symmetry breaking in the vacuum sector. Hence, adopting the arguments in [7], the particle nature of the Goldstone modes in thermal equilibrium states may be regarded as settled.

It is easily seen in examples that the above mentioned properties of the damping factors do not imply that there holds a sharp energy–momentum dispersion law for the Goldstone particles. This fact corroborates the statement made in [7] that the Källén–Lehmann type representation (2) is better suited than Fourier analysis to uncover the particle aspects of thermal equilibrium states. The representation is also useful for the analysis of the more subtle momentum space properties of thermal correlation functions, where one has to employ techniques of complex analysis. We intend to return to this interesting issue elsewhere.

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